*On Certain Saltus Equations.**

By Henry Blumberg.

Introduction.

Instead of attempting to introduce into our discussion the utmost generality, we shall, for the sake of greater simplicity, confine ourselves to the consideration of a real, one-valued function $f(x)$, defined in the linear continuum, bounded at every point—and hence, according to the Borel theorem on sets of intervals, in every interval—and unrestricted as to continuity. From $f(x)$ we derive three new functions: $u(f, x)$, the upper-bound (=maximum) function; $l(f, x)$, the lower-bound (=minimum) function; and $s(f, x)=u(f, x)-l(f, x)$, the saltus (=oscillation) function of f.† Just as the notion of derivative at once suggests that of differential equation, so the notion of saltus leads to that of "saltus equation." It is the principal object of the present paper to give what may be regarded as complete solutions of such saltus equations in several simple cases. For the sake of greater brevity, we write as follows the successive saltus functions derived from $f(x)$:

$$s(f, x)=s'_f(x), \quad s(s'_f, x)=s''_f(x), \quad s(s''_f, x)=s'''_f(x), \; \ldots .$$

Because of a theorem due to Sierpiński,‡ which asserts that $s'''_f(x)=s''_f(x)$ no matter what function $f(x)$ we start with, there is no need of considering saltus equations beyond the "second order," *i. e.*, equations involving a saltus with an index greater than 2. The saltus equations $s'_f(x)=g(x)f(x)$ and $s''_f(x)=g(x)s'_f(x)$, where $g(x)$ is an arbitrarily given continuous function and $f(x)$ is sought, are among those above referred to as "completely solved" in this paper.

The first section deals with several properties of functions of interest in themselves and useful later. Section 2 deals with the equation $s'_f(x)=g(x)f(x)$; Section 3, with the equation $s''_f(x)=g(x)s'_f(x)$; and the appendix indicates several lines of generalization.

* Read before the American Mathematical Society, December 26, 1913.

† For the definitions of these functions see the author's paper, "On Certain General Properties of Functions," *Annals of Mathematics*, Vol. XVIII (1917), p. 147, and Hobson, "The Theory of Functions of a Real Variable" (1907), Art. 180.

‡ *Bulletin de l'Académie des Sciences de Cracovie* (1910), pp. 633–634.

Section 1. *Preliminary Theorems and Several Simple Saltus Equations.*

THEOREM I. *If two given functions are upper-semi-continuous (lower-semi-continuous) at a given point ξ, the saltus of their sum at ξ is greater than or equal to the saltus of each of the given functions at ξ.*

For let $f_1(x)$ and $f_2(x)$ be upper-semi-continuous at ξ. This condition is equivalent to the relations $f_1(\xi)=u(f_1,\xi)$ and $f_2(\xi)=u(f_2,\xi)$, where $u(f_1,x)$ and $u(f_2,x)$ are the upper-bound functions of f_1 and f_2, respectively. If $s'_{f_1}(\xi)=h$, a sequence $\{\xi_n\}$ of real numbers exists such that $\lim_{n\to\infty}\xi_n=\xi$ and $\lim_{n\to\infty} f_1(\xi_n)=u(f_1,\xi)-h=f_1(\xi)-h$. Moreover, since $f_2(\xi)$ is upper-semi-continuous, we have $\lim_{n\to\infty} f_2(\xi_n)\leq f_2(\xi)$. Hence

$$\lim_{n\to\infty}\,[f_1(\xi_n)+f_2(\xi_n)]\leqq f_1(\xi)+f_2(\xi)-h,$$

which shows that the function $f(x)=f_1(x)+f_2(x)$ has a saltus $\geqq h$ at ξ. In the same way, we prove the theorem for $f_2(x)$, and for two lower-semi-continuous functions.

Since the saltus function $s'_f(x)=u(f,x)-l(f,x)$ is actually the sum of two upper-semi-continuous functions, $u(f,x)$ and $-l(f,x)$, we have, as an application of Theorem I,

THEOREM II. *The second saltus function is at every point greater than or equal to the saltus of both the upper-bound function and the lower-bound function.*

If in Theorem I it happens that the saltus of the sum is zero at ξ, then it follows that the saltus of both f_1 and f_2 is zero at ξ. That is, we have

THEOREM III. *If the sum of two given upper-semi-continuous functions is continuous at a point ξ, then each of the given functions is continuous as ξ.**

Either from Theorem II or Theorem III, we obtain by specialization

THEOREM IV. *If $s'_f(x)$ is continuous at a given point ξ, both $u(f,x)$ and $l(f,x)$ are continuous at ξ.*†

By means of Theorem IV we can obtain the complete solution of the saltus equation

$$s''_f(x)=0.$$

For from this equation it follows that $s'_f(x)$ is continuous, and hence, according to Theorem IV, both $u(f,x)$ and $l(f,x)$ are continuous functions. Conversely,

*The theorem is false for the *product* of two upper-semi-continuous functions.

†Although this theorem is near at hand and of sufficient interest to deserve mention among the general properties of functions, it does not seem to have been previously noted.

if $f(x)$ is such that $u(f, x)$ and $l(f, x)$ are continuous, it follows that $s'_f(x)$ is continuous, and hence $s''_f(x)=0$. Therefore, if $f_1(x)$ and $f_2(x)$ are any two continuous functions whatever, and we take the function $f(x)$ such that the "curve" whose equation is $y=f(x)$ lies everywhere between the curves $y=f_1(x)$ and $y=f_2(x)$ and has every point of each of these curves as a limiting point, then $f(x)$ is a solution of the saltus equation $s''_f(x)=0$; and every solution is so obtainable.

DEFINITION. The function $f(x)$ is said to be "*continuously bounded*" in a given interval (linear continuum) if $u(f, x)$ and $l(f, x)$ are continuous in the interval (linear continuum).*

We now have

THEOREM V. *A necessary and sufficient condition for a solution of the saltus equation $s''_f(x)=0$ is that it shall be continuously bounded.*

REMARK. *The saltus equation $s''_f(x)=g(x)$, where $g(x)$ is an arbitrarily given continuous function of x, is only apparently less restrictive upon $f(x)$ than the equation $s''_f(x)=0$ of Theorem V.* For since $s''_f(x)$ is zero in an everywhere dense set,† its continuity implies that it is identically zero.

DEFINITION. A function is said to be "*pointwise discontinuous at a given point ξ,*" if it possesses at least one point of continuity in every neighborhood of ξ. ‡

THEOREM VI. *The relation*

$$s''_f(\xi)=s'_f(\xi)$$

is a necessary and sufficient condition for the pointwise discontinuity of f at ξ.

For on the one hand, let $f(x)$ be pointwise discontinuous at ξ. Then every neighborhood of ξ contains at least one point where f is continuous, *i. e.*, where $s'_f(x)=0$. Therefore $l(s'_f, \xi)=0$. Since $s'_f(x)$ is upper-semi-continuous, we have $u(s'_f, \xi)=s'_f(\xi)$. Consequently

$$s''_f(\xi)=u(s'_f, \xi)-l(s'_f, \xi)=s'_f(\xi).$$

On the other hand, suppose $f(x)$ is not pointwise discontinuous at ξ. Then a neighborhood (α, β) of ξ exists such that f is totally discontinuous in (α, β), hence $s'_f(x)>0$ for $\alpha\leqq x\leqq\beta$. It follows, since an upper-semi-continuous function and hence $s'_f(x)$ attains its (greatest) lower bound in a closed interval,

* Virtually all the definitions and theorems of this paper apply equally well to an interval and to the entire linear continuum. Further specific reference to this fact is regarded as unnecessary.

† *Annals of Mathematics, loc. cit.*, p. 151.

‡ Continuity at ξ implies pointwise discontinuity at ξ. We use "pointwise discontinuous" in the sense of "at most pointwise discontinuous," thus deviating somewhat from the more common usage for the sake of greater convenience.

that this lower bound of $s'_f(x)$ for $\alpha \leqq x \leqq \beta$ is a positive number, say h. Since, on account of the upper-semi-continuity of $s'_f(x)$, the relation $u(s'_f, \xi) = s'_f(\xi)$ holds, we have

$$s''_f(\xi) = u(s'_f, \xi) - l(s'_f, \xi) = s'_f(\xi) - l(s'_f, \xi) < s'_f(\xi) - h < s'_f(\xi).$$

Thus $s''_f(\xi) \neq s'_f(\xi)$ is a consequence of the fact that f is not pointwise discontinuous at ξ, and Theorem VI is completely proved.

From Theorem VI we obtain

THEOREM VII. *The set of solutions of the saltus equation* $s''_f(x) = s'_f(x)$ *is identical with the set of pointwise discontinuous functions.*

THEOREM VIII. *A necessary and sufficient condition that* $f(x)$ *be a solution of the saltus equation* $s'_f(x) = f(x)$ *is that it possesses the following two properties:** (α) f *is upper-semi-continuous;* (β) $f = 0$ *in an everywhere dense set.*

In the first place, the two properties are necessary. For (α) follows from the upper-semi-continuity of $s'_f(x)$. Since (α) implies that $u(f, x) = f(x)$, we have $s'_f(x) = u(f, x) - l(f, x) = f(x) - l(f, x)$ and therefore we must have $l(f, x) = 0$ to secure the relation $s'_f(x) = f(x)$. But $f(x)$, being upper-semi-continuous, attains its (greatest) lower bound, that is 0, in every closed interval; and hence (β) holds. In the second place, the properties are sufficient. For from (α), we conclude that $u(f, x) = f(x)$, and from (β) and (α), that $l(f, x) = 0$, whence $s'_f(x) = f(x)$.

Section 2. *The Saltus Equation* $s'_f(x) = g(x) f(x)$, *where* $g(x)$ *is a Continuous Function.*

(a) *If* $g(\xi) \neq 0, 1$ *we have* $f(\xi) = 0$. For let (α, β) be an interval containing ξ in its interior and such that $g(x) \neq 0, 1$ for $\alpha \leqq x \leqq \beta$. We may then write $f(x) = s'_f(x)/g(x)$, $\alpha \leqq x \leqq \beta$, which shows, in virtue of the upper-semi-continuity of $s'_f(x)$ that $f(x)$ is upper-semi-continuous in (α, β). $s'_f(x)$ being zero at every point of continuity of $f(x)$, must therefore vanish in an everywhere dense set S of (α, β). Consequently f vanishes in S, and therefore in virtue of Theorem VIII we have $s'_f(x) = f(x)$, $\alpha < x < \beta$. But the relation $s'_f(x) = f(x)$ would imply that $g(x) = 1$, contrary to our assumption, unless $f(x) = 0$. In particular, therefore, $f(\xi) = 0$.

(b) *If* $g(\xi) = 1$, *then* $f(x)$ *is upper-semi-continuous at* ξ, *and every neighborhood of* ξ *contains a zero of* $f(x)$. For consider the two possibilities: (1) ξ is an interior point of an interval (α, β) where $g(x)$ is constantly equal

* Cf. the author's paper, *Annals of Mathematics, loc. cit.*, p. 151.

to 1; (2) ξ is not in such an interval. If (1) holds, we have $s'_f(x)=f(x)$ for $\alpha<x<\beta$, and the assertions in question follow from Theorem VIII. If (2) holds, there are points where $g(x)\neq 0, 1$ in every neighborhood of ξ; hence, according to (a), every neighborhood of ξ contains zeros of $f(x)$. Consequently, since in virtue of our saltus equation a neighborhood of ξ exists in which $f(x)\geqq 0$, we have $l(f,\xi)=0$, whence $s'_f(\xi)=u(f,\xi)$. Therefore $u(f,\xi)=f(\xi)$, which is equivalent with the upper-semi-continuity of f at ξ.

(c) *If* $g(x)=0$, $f(x)$ *is continuous at* ξ.

(a), (b) and (c) give us necessary relations between the character of the given function $g(x)$ and a solution of our saltus equation. Furthermore, however, these relations completely characterize the solutions. For suppose $f(x)$ satisfies these relations. In the first place, let ξ be such that $g(\xi)\neq 0, 1$. Then ξ is contained in an interval throughout which $g(x)\neq 0, 1$. According to (a), $f(x)=0$ throughout this interval, so that $s'_f(\xi)=0$, $s'_f(\xi)=g(\xi)f(\xi)$. In the second place, let $g(\xi)=1$. Then, as an easy consequence of (a) and (b), we have $l(f,\xi)=0$. This relation, taken in conjunction with the upper-semi-continuity of f, shows that $s'_f(\xi)=f(\xi)=g(\xi)f(\xi)$. In the third place, let $g(\xi)=0$. Then f is continuous according to (c), and hence $s'_f(\xi)=0=g(\xi)f(\xi)$. The saltus equation is thus satisfied at every point, and we have

THEOREM IX. *If* $g(x)$ *is a given continuous function, then the set of solutions of the saltus equation* $s'_f(x)=g(x)f(x)$ *is identical with the set of functions* f *such that* (a) $f(x)=0$ *where* $g(x)\neq 0,1$; (b) $f(x)$ *is upper-semi-continuous where* $g(x)=1$ *and* x *does not lie in the interior of an interval throughout which* $g(x)=1$; (c) $f(x)$ *is upper-semi-continuous and possesses an everywhere dense set of zeros in the interior of every interval throughout which* $g(x)=1$; *and* (d) $f(x)$ *is continuous where* $g(x)=0$.

The following implications of Theorem IX deserve mention.

THEOREM X. *The only solution of the saltus equation* $s'_f(x)=g(x)f(x)$, *where* $g(x)$ *is a given continuous function of* x *taking nowhere the values* 0, 1, *is* $f(x)\equiv 0$.

THEOREM XI. *The only solution of the saltus equation* $s'_f(x)=g(x)f(x)$, *where* $g(x)$ *is a given continuous function taking at no point the value* 1, *and throughout no interval the value* 0, *is* $f(x)\equiv 0$.

THEOREM XII. *A necessary and sufficient condition that* $f(x)$ *be a solution of the saltus equation* $s'_f(x)=g(x)f(x)$, *where* $g(x)$ *is a continuous function taking nowhere the value* 1, *is that* $f(x)$ *be continuous, and* $=0$ *where* $g(x)\neq 0$.

THEOREM XIII. *If no interval exists throughout which the continuous function $g(x)$ is 0 or 1, then all the solutions of the saltus equation $s'_f(x) = g(x) f(x)$ may be obtained by making $f(x)$ an arbitrary non-negative, upper-semi-continuous function in the set of points where $g(x) = 1$ and giving it the value 0 elsewhere.**

Section 3. The Saltus Equation $s''_f(x) = g(x) s'_f(x)$, where $g(x)$ is a Continuous Function.

(a) *If $g(\xi) \neq 0, 1$, then $f(x)$ is continuous at ξ.* For an interval (α, β) with ξ as interior point exists, such that $g(x) \neq 0, 1$ for $\alpha \leqq x \leqq \beta$. Since $s''_f(x)$ is 0 in an everywhere dense set, it follows from our saltus equation that $s'_f(x) = 0$ is an everywhere dense subset of (α, β). We now see that $s'_f(x)$ possesses in (α, β) the properties sufficient, according to Theorem VIII, to make it a solution of the saltus equation $s'_F(x) = F(x)$. Hence $s(s'_f, x) = s'_f(x)$; *i. e.*, $s''_f(x) = s'_f(x)$ in (α, β). In particular, $s''_f(\xi) = s'_f(\xi)$, from which we would conclude, contrary to our assumption, that $g(\xi) = 1$, unless $s'_f(\xi) = 0$. Accordingly $f(x)$ is continuous at ξ.

(b) *If $g(\xi) = 0$ and ξ is not an interior point of an interval where $g(x)$ is constantly 0, then $f(x)$ is continuous at ξ.* For since $s''_f(\xi) = 0$, it follows that $s'_f(x)$ is continuous at ξ. As every neighborhood of ξ contains points where $g(x) \neq 0, 1$, we conclude from (a) that every neighborhood of ξ contains points where $s'_f(x) = 0$. Hence $s'_f(\xi) = 0$, and $f(x)$ is continuous at ξ.

(c) *If $g(x) = 0$ throughout the interior of an interval (α, β), then $f(x)$ is continuously bounded for $\alpha < x < \beta$* (Theorem V).

(d) *If $g(\xi) = 1$, then $f(x)$ is pointwise discontinuous at ξ* (Theorem VI).

Just as in the preceding section, the necessary conditions (a), (b), (c) and (d) upon a solution $f(x)$ of our present saltus equation are sufficient to characterize it completely. For suppose $f(x)$ satisfies these conditions. In the first place, let $g(\xi) \neq 0, 1$; then $f(x)$ is continuous at ξ and hence $s'_f(\xi) = 0$. Since $s'_f(x)$ is upper-semi-continuous and non-negative, we have $s''_f(\xi) \leqq s'_f(\xi)$. Therefore $s''_f(\xi) = 0$ and the equation $s''_f(x) = g(x) s'_f(x)$ is satisfied for $x = \xi$. In the second place, let $g(\xi) = 0$ and let furthermore ξ be such that it is not an interior point of an interval where $g(x)$ is constantly 0; then $f(x)$ is continuous at ξ. Hence $s'_f(\xi) = 0$, and therefore $s''_f(\xi) = 0$, and $s''_f(\xi) = g(\xi) s'_f(\xi)$. That (c) and (d) are sufficient conditions follows from Theorems V and VI, respectively. We thus have

* The notion here employed of an upper-semi-continuous function defined in an arbitrary linear point-set requires no further explanation.

THEOREM XIV. *The set of solutions of the saltus equation* $s_f''(x) = g(x)s_f'(x)$, *where* $g(x)$ *is a given continuous function, is identical with the set of functions that are pointwise discontinuous at every point where* $g(x)=1$, *continuously bounded in the interior of every interval where* $g(x)$ *is constantly* 0 *and elsewhere continuous.**

A number of consequences of Theorem XIV deserve special mention:

THEOREM XV. *If the continuous function* $g(x)$ *is* 1 *nowhere and* 0 *throughout no interval, then the saltus equation* $s_f''(x)=g(x)s_f'(x)$ *has only the trivial solution* $f(x)=$ *continuous function.*

THEOREM XVI. *If the continuous function* $g(x)$ *is nowhere* $=1$, *the set of solutions of the saltus equation* $s_f''(x)=g(x)s_f'(x)$ *is identical with the set of continuously bounded functions that are continuous where* $g(x)\neq 0$.

THEOREM XVII. *If there is no interval throughout which the continuous function* $g(x)$ *is* 0 *or* 1, *then all the solutions of the saltus equations* $s_f''(x) = g(x)s_f'(x)$ *are obtained by assigning arbitrary values to* $f(x)$ *at the points where* $g(x)=1$ *and making* $f(x)$ *continuous elsewhere.*

Appendix.

A. The coefficients in the saltus equations considered in this paper have so far been exclusively continuous functions. Without attempting a comprehensive treatment for the case of discontinuous coefficients, we shall now give a theorem relating to the saltus equation $s_f''(x)=g(x)s_f'(x)$ for a discontinuous g.

THEOREM XVIII. *If the zeros of* $g(x)$ *form (at most) a non-residual set*† *in every interval, then the set of solutions of the saltus equation* $s_f''(x)=g(x)s_f'(x)$ *is identical with the set of pointwise discontinuous functions that are continuous where* $g(x)\neq 1$.

For since $s_f'(x)$ is upper-semi-continuous, it is continuous in a residual set,‡ hence $s_f''(x)=0$ in a residual set. Therefore, according to the equation $s_f''(x)=g(x)s_f'(x)$, the product $g(x)s_f'(x)$ is equal to zero in a residual set. By hypothesis, the zeros of $g(x)$ constitute a non-residual set in every interval;

*A part of the content of Theorem XIV may be derived as follows: Taking the saltus of each side of the equation $s''_f(x)=g(x)s'_f(x)$, we obtain the equation $s'''_f(x)=g(x)s''_f(x)$. But according to Sierpiński's theorem, $s'''_f(x)\equiv s''_f(x)$. Therefore either $s''_f(x)=0$ or $g(x)=1$. In similar fashion, we may deal with the saltus equation of Section 2.

† According to Denjoy, an "*exhaustible*" set ($=$ set of first category) is the sum of a denumerable set of non-dense sets; and a "*residual*" set, the complementary set of an exhaustible set. See *Journal de Mathématiques*, Sér. 7, Vol. I (1915), pp. 122–125.

‡ See, for example, Hobson, "The Theory of Functions of a Real Variable" (1907), p. 240.

consequently, the zeros of $s'_f(x)$ constitute a non-exhaustible set in every interval. The zeros of $s'_f(x)$ are therefore everywhere dense. Since additionally $s'_f(x)$ is upper-semi-continuous, we conclude from Theorem VIII that $s''_f(x) = s(s'_f, x) = s'_f(x)$, which shows according to Theorem VII that f must be pointwise discontinuous. Furthermore, we now have two possibilities: (1) Either $s'_f(x) = 0$, and then f is continuous at x. Or (2) $s'_f(x) \neq 0$, and then $g(x) = s''_f(x)/s'_f(x) = 1$.

We have thus shown that f must necessarily be pointwise discontinuous, and continuous where $g(x) \neq 1$. But these conditions are also sufficient. For from the pointwise discontinuity of f, it follows that $s''_f(x) = s'_f(x)$, and hence our saltus equation is satisfied where $g(x) = 1$; and from the continuity of f at other points, it follows that the saltus equation is also satisfied elsewhere.

One of the consequences of Theorem XVIII is

THEOREM XIX. *The only solutions of the saltus equation* $s''_f(x) = g(x) s'_f(x)$, *where* $g(x)$ *is a given function taking the value* 1 *nowhere and the value* 0 *in a non-residual set in every interval, are continuous functions.*

B. Although we have assumed throughout the paper that $f(x)$ is single-valued and bounded at every point, it may be readily seen that the dropping of the former assumption would necessitate hardly any change in our presentation, while the dropping of the latter would require only slight modification of the results and proofs. The extension of the treatment to functions of more than one variable is not difficult, but we shall not enter upon it here. We shall also not deal with the corresponding saltus equations that arise when the f-saltus, the e-saltus, the z-saltus, etc.,* are employed instead of the ordinary saltus.

C. In conclusion we call attention to the simple saltus equation $s'_f(x) = g(x)$, where $g(x)$ is an arbitrary discontinuous function. We have not succeeded in obtaining a satisfactory characterization of its solutions.

* *Annals of Mathematics, loc. cit.*, pp. 148 and 149.